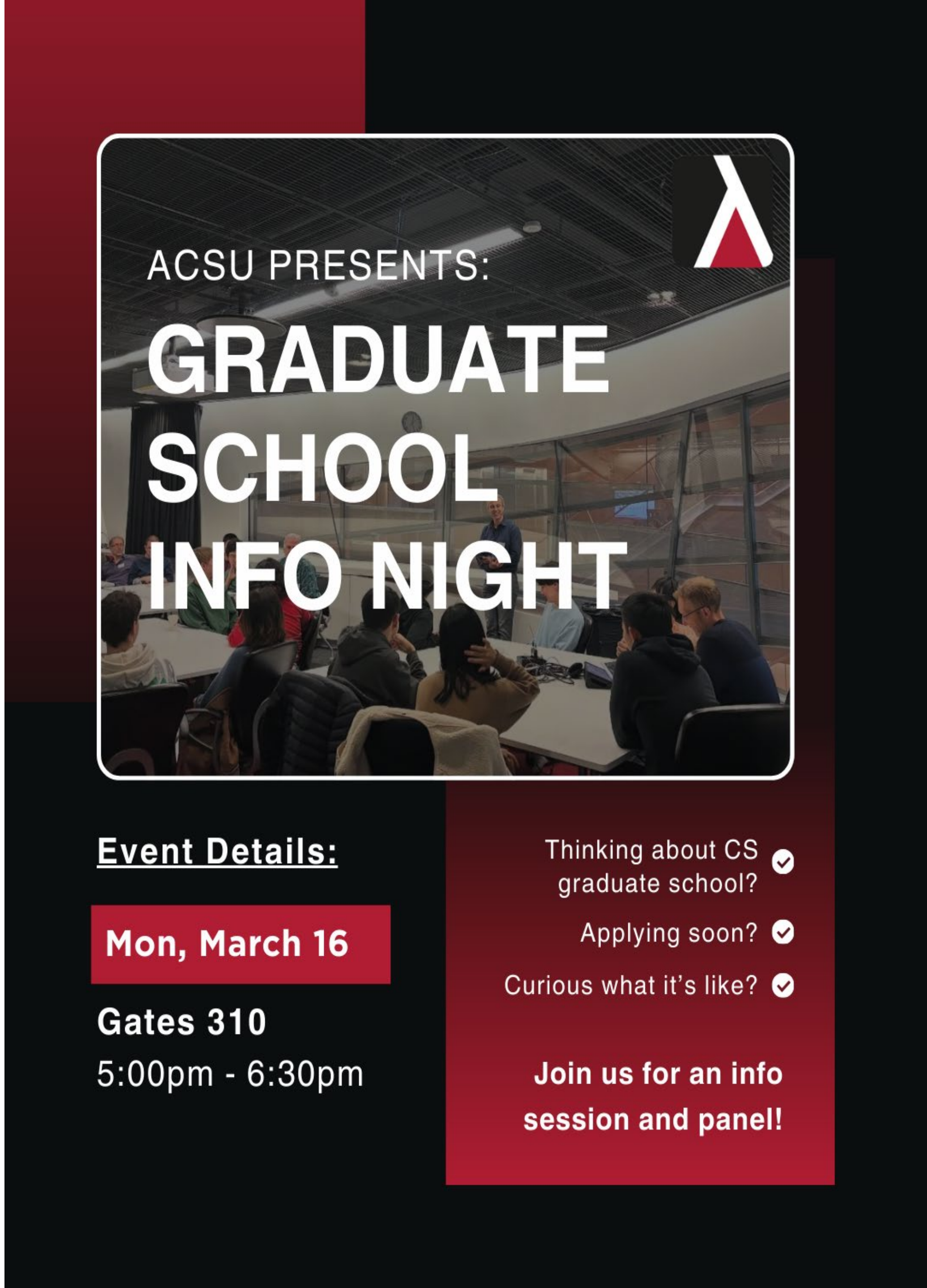
Announcements

- HW7 single question, due Tuesday March 17th max 1 late day
- Prelim 2: Tuesday March 24th
  - The conflicts survey is open, due today
  - Topics: stable matching, flows and applications and NP-completeness
    ending with today's section
  - Information sheet on topics and sample question in canvas

Thank you for all who responded to the survey (147)

- office hours too crowded
- easier/harder/more
- request to review



Divide and Conquer: first example Merge Sort

Recall merge-sort given $\underbrace{x_1 \ldots}\ \underbrace{x_n}$ numbers

Merge sort $(x_1 \ldots x_n)$

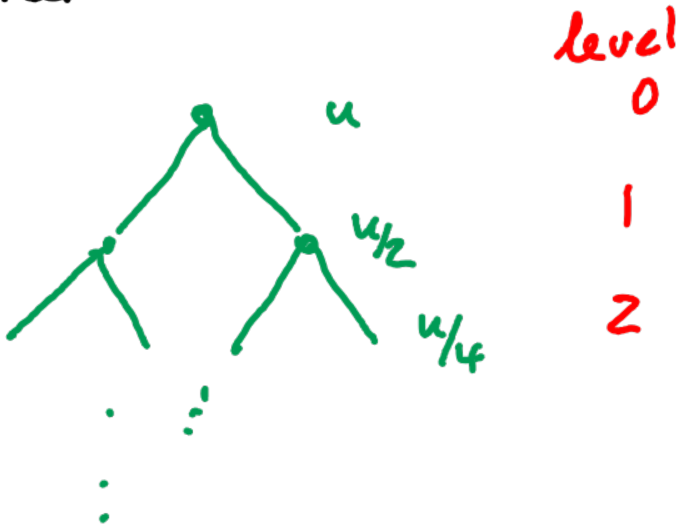
- Mergesort $(x_1 \ldots x_{n/2})$
- Mergesort $(x_{n/2+1} \ldots x_n)$
- merge two parts

sorted sorted

Running time: $T(n)$ time for list of size n

$$T(n) \leq 2T(n/2) + c \cdot n$$

Note: merge two sorted lists is $O(n)$



- # levels: $\log_2 n$
- # subproblems at level i $\quad 2^i$
- size of problems level i $\quad n/2^i$

$\Rightarrow$ time spent at level i $\quad c \cdot n$

$O(n \log n) \Leftarrow$

Topic today: integer multiplication

very long integer, say n digits

```
 2435 * 235
 4870
  7305
  12175
 572225
```

elementary school method
$O(n^2)$ time

Recursive divide & conquer

$x = \overbrace{\quad}^{x_1}\overbrace{\quad}^{x_2} \quad * \quad \overbrace{\quad}^{y_1}\overbrace{\quad}^{y_2} = y$

$x = x_1 10^{n/2} + x_2$ $\qquad$ $y = y_1 10^{n/2} + y_2$

$$xy = (x_1 10^{n/2} + x_2)(y_1 10^{n/2} + y_2) =$$
$$= x_1 y_1 10^n + (x_1 y_2 + x_2 y_1) 10^{n/2} + x_2 y_2$$

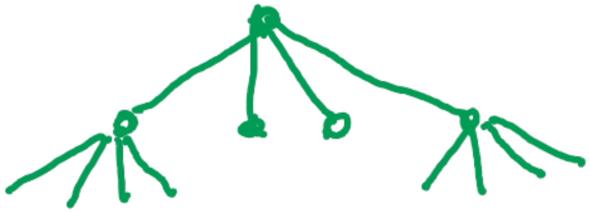
Algorithm: recursively compute $x_1 y_1$, $x_1 y_2$, $x_2 y_1$, $x_2 y_2$
& add them up (note multiplying with 10^k easy)

time for addition:

$O(n)$ for n digit numbers

$$T(n) \le 4\,T(n/2) + cn$$

tree of recursive calls



level	# problems	size of problem
0	1	n
1	4	$n/2$
$\vdots$		
i	4^i	$n/2^i$

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What is the running time of the recursive integer multiplication defined above on n digit integers

$T(n) \le 4T(n/2) + c \cdot n$

A. $O(n)$
B. $O(n \log n)$
C. $O(n^2)$
D. None of these
E. I don't know

level	#	size
0	1	n
1	4	$n/2$
2	16	$n/4$
i	4^i	$n/2^i$

\# levels $\log_2 n$

$c \cdot n + 4 \cdot c\, n/2 + 16 \cdot c \cdot n/4 + \dots 4^i c \cdot n/2^i + \dots + 4^{\log_2 n} c \frac{n}{2^{\log_2 n}}$

time spent per level

$= c \cdot n + 2\, c \cdot n + 4\, c \cdot n + \dots 2^i c \cdot n + \dots + \overbrace{2^{\log_2 n} c \cdot n}^{c \cdot n^2} = c \cdot n^2 + c\, n^2/2 + \dots + c\, n^2/2^{\cdot} \le 2 c\, n^2$

Faster: Recursive integer multiplication (Karatsuba)

$$xy = x_1 y_1 10^n + \underbrace{(x_1 y_2 + x_2 y_1)} 10^{n/2} + x_2 y_2$$

Algorithm:

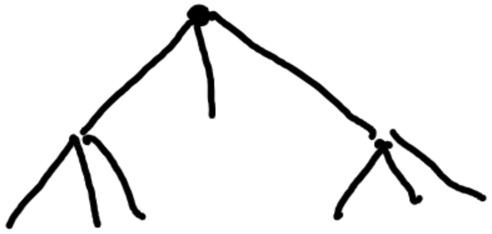
- compute $x_1 \cdot y_1$
- comput $x_2 \cdot y_2$
- compute $(x_1 + x_2) \cdot (y_1 + y_2) = res = x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2$
- compute $x_1 y_2 + x_2 y_1 = res - x_1 y_1 - x_2 y_2$

$$T(n) \leq 3T(n/2) + cn$$

(addition & subtraction linear time)

Assume n is a power of 2

Analyzing Karatsuba's algorithm's running time



level	size	# problems
0	n	1
1	$n/2$	3
2	$n/4$	9
i	$n/2^i$	3^i

time spent at level i

$$3^i \cdot c \cdot n/2^i = c\left(\frac{3}{2}\right)^i \cdot n$$

levels: $\log_2 n$

total time: $$\sum_{i=0}^{\log_2 n} c \cdot \left(\frac{3}{2}\right)^i \cdot n = c \cdot n \cdot \frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{1/2} \leq 2c\cancel{n} \cdot \frac{3^{\log_2 n}}{\cancel{2^{\log_2 n}}}$$

$$= 2c\, 3^{\log_2 n} = 2c \cdot n^{\log_2 3}$$

$$= O(n^{1.58\ldots})$$

Recall $$\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}$$

$$a^{\log_2 n} = n^{\log_2 a}$$

take log $$\log_2 n \log_2 a = \log_2 n \cdot \log_2 a$$